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DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

253. Proposed by PROF. R. D. CARMICHAEL, Anniston, Ala.

Prove that $x^5+ax+b=0$ is solvable by radicals if $b=ma$, m being the negative of half the sum of any two roots of the original equation. Exhibit the solution.

Solution by DR. L. E. DICKSON, The University of Chicago.

Let r and s be the two roots for which $m=-\frac{1}{2}(r+s)$. Then

$$r^5+ar+ma=0,\ s^5+as+ma=0.$$

Adding, we get $r^5+s^5=0$. We exclude the trivial case $m=0$, so that $r+s\neq 0$, $r=-es$, e being an imaginary fifth root of unity. From $r+es=0$, $r+s=-2m$, we get

$$(1)\qquad s=\frac{2m}{e-1},\ r=\frac{-2me}{e-1}.$$

Now s will indeed be a root if and only if

$$32m^4+al=0,\ l=(e+1)(e-1)^4.$$

Since $e^4+e^3+e^2+e+1=0$, we get $l=5(e^3+e^2+1)$. Now e^2+e^3 and $e+e^4$ are the roots $\frac{1}{2}(-1\pm\sqrt{5})$ of $z^2+z-1=0$. Hence the problem stated is possible if and only if

$$64m^4+5(1\pm\sqrt{5})a=0,\ a=\tfrac{16}{5}(1\mp\sqrt{5})m^4.$$

For m arbitrary, and for this value of a, the equation $x^5+ax+ma=0$ has the two roots (1); the remaining roots may be found by solving a cubic.

284. Proposed by DR. E. H. MOORE, The University of Chicago, Chicago, Ill.

Discuss the system of equations:

$$\begin{cases} x^k+y^k=a_k \\ x^l+y^l=a_l \quad (k,\ l \text{ distinct positive integers}) \end{cases}$$

in general and for particular values of $(k, l;\ a_k,\ a_l)$.

Solution by BENJAMIN F. FINKEL, Ph. D., Drury College, Springfield, Mo.

§1. GENERAL DISCUSSION.

Assume $k<l$. Solve the first equation for y, say, and we have $y=(a_k-x^k)^{1/k}$. Substitute this value of y in the second equation and free the resulting equation from fractional exponents. The equation thus obtained is called the *Resultant Equation** in x, and is, at most, of the klth degree. It has, therefore, kl roots, real, imaginary, or infinite.

If k and l are both odd or both even, the resultant equation is of the $kl-k$th degree and there are kl roots, k of which are infinite, and, therefore, $kl-k$ roots, real or imaginary.

If k is odd and l even, or if k is even and l odd, the resultant equation is of the klth degree and there are kl roots.

Since the equations are symmetric in x and y the resultant equation for y is the same as that for x. Hence x and y have the same series of values, but not every value of y can be taken with every value of x and satisfy the given system of equations. The final test of the legitimacy of every value of x and y is that these values when substituted in the given system of equations shall satisfy them, and every such set of values of x and y constitutes a solution.

Since the two equations of the system are symmetric with respect to x and y, it follows that if $(x, y)\equiv(a, b)$, $a\neq b$, is a solution, then $(x, y)\equiv(b, a)$ is also a solution. Since, as we have seen above, the greatest number of sets of values of x and y satisfying the given equations is kl, it follows that these sets are made up of sets of values of x and y in which x and y have different values, the number of such sets not exceeding $\frac{1}{2}kl$, and sets of values of x and y in which x and y have the same value.

Let us assume that there are kl solutions of the given system of equations and let $2n'$ be the number of sets of values of x and y wherein x and y have different values, *e. g.*, $(x,y)\equiv(a, b)$, $a\neq b$, and let n'' be the number of sets wherein x and y have the same values, *e. g.*, $(x, y)\equiv(c, c)$.

Then we must have

$$2n'+n''=kl.$$

kl is of the form $2m$ or $2m+1$.

(1) Assume kl is of the form $2m$. Then we have $2n'+n''=2m$, the solutions of which are

n'	n''
m	0
$m-1$	2
$m-2$	4
$\vdots$	$\vdots$
0	$2m$

---

*Chrystal's *Algebra*, Part I, p. 403, ed. 1885.

(2) Assume kl of the form $2m+1$. Then we have $2n'+n''=2m+1$, the solutions of which are

n'	n''
m	1
$m-1$	3
$m-2$	5
⋮	⋮
0	$2m+1$

Hence, it follows that when kl is even the number of solutions in which x and y are equal must be even, and when kl is odd, the number of such solutions must be odd.

The graph of the equation $x^n+y^n=a_n$, when n is even and greater than 2 and $a_n>0$ is of the form of Fig. 1. When $n=2$ and $a_n>0$, the graph is a circle.

When n approaches an infinitely large even number and a_n is a finite positive number, the graph approaches a square whose sides are 2. When $a_n<0$, the graph is imaginary.

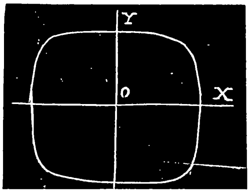

Fig. 1.

The graph of the equation $x^n+y^n=a_n$, when n is odd and greater than 1 is of the form of Fig. 2. When n is 1, the graph is a straight line. When n approaches an infinitely large positive odd integer and a_n is a finite positive number, the graph approaches the full-line curve in Fig. 3. When n approaches an infinitely large odd integer and $a_n<0$, the graph approaches the dotted line in Fig. 3.

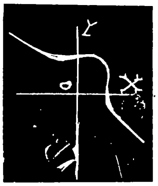

Fig. 2.

If, instead of solving one of the equations with respect to one of the unknowns and substituting this value in the second, to obtain the resultant equation in the other unknown, one solves the two equations for the ratio, r, of x to y, there results, in general, an equation of the klth degree in r. And since the two equations are symmetric in x and y, this equation in r will be a *reciprocal equation.* This fact often enables one to obtain solutions more easily than by the first method. It also enables one to pair the proper values of x and y. It does not, however, prevent the introduction of extraneous roots. Thus, after r is found, and one wishes to find x, say, there will be for each of the kl values of r as many values of x as there are units in the exponent of x in the equation in which r is substituted. Thus, suppose $x^k(1+r^k)=a_k$; from this equation one would obtain k values of x for each of the kl values of r. One would thus obtain k^2l values in all for x, of which not more than kl can be legitimate values, that is, values which when used with certain values of y will satisfy the system of equations.

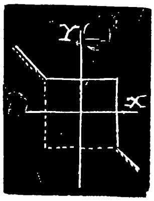

Fig. 3.

§2. SPECIAL CASES.

a. If $k=l$ and $a_k=a_l$, then there are an infinitude of solutions. Geometrically interpreted, the two equations represent two coincident curves. Hence, the co-ordinates of all points of the curves will satisfy the two equations.

If $k=l$ and $a_k \neq a_l$, the roots are all infinite (real or imaginary). Thus, let $k=l=1$ and, therefore, $x+y=a$ and $x+y=b$, $a \neq b$. These equations are frequently called inconsistent, but for the sake of uniformity of language they are commonly said to be satisfied by the point at infinity. $(x, y) \equiv (\infty, -\infty)$, $(-\infty, \infty)$ are solutions. It thus appears that there are two solutions here. We may, however, consider this to be really only one solution; for geometrical considerations have led us to regard $-\infty$ and ∞ as the same point if the real number system be represented by points in a straight line.* The equations represent two parallel straight lines.

Suppose $x^2+y^2=a^2$ and $x^2+y^2=b^2$, $a \neq b$. Here, $x=\pm\infty$ and hence $y=\pm\infty i$. Since x and y are symmetric we have 2^3 solutions. These are, $(\infty, \infty i)$, $(\infty, -\infty i)$, $(-\infty, \infty i)$, $(-\infty, -\infty i)$, $(\infty i, \infty)$, $(\infty i, -\infty)$, $(-\infty i, \infty)$, and $(-\infty i, -\infty)$. But from the previous considerations, these eight solutions may be reduced to four, the requisite number; for we may consider $(\infty, \infty i)$, $(\infty, -\infty i)$ as one point, and $(-\infty, \infty i)$, $(-\infty, -\infty i)$ as a second point and so on. If we assume that parallel lines in space intersect at infinity, these four solutions may be reduced to two solutions, the first four above constituting one solution and the last four a second solution.

By a more generally accepted convention, we may consider the two solutious as the "circular points at infinity," through which two points every circle passes. The co-ordinates of one of the circular points at infinity may be considered to be $(\infty, \infty i)$ or $(-\infty, -\infty i)$; these co-ordinates, by the previous consideration, being one and the same point, and the co-ordinates of the other point at infinity being $(\infty, -\infty i)$ or $(-\infty, \infty i)$. These points may be constructed geometrically by the method suggested in Carr's *Synopsis of Pure and Applied Mathematics*, pp. 674-677.

Suppose $x^3+y^3=a$, $x^3+y^3=b$, $a \neq b$. Here, $x=\infty$, $\omega\infty$, $\omega^2\infty$, and $y=-\infty$, $-\omega\infty$, $-\omega^2\infty$. There are here eighteen algebraic solutions, but these, as before, may be reduced to nine, the requisite number.

Suppose $x^4+y^4=a$, $x^4+y^4=b$, $a \neq b$. Here, $x=\pm\infty$, $\pm\infty i$, and $y=\pm\infty\sqrt{i}$, $\pm\infty i^{\frac{3}{2}}$. There are here sixteen solutions, the requisite number.

b. Suppose $k=1$, $l=2$, and, therefore, $kl=2$. Here, $2n'+n''=2$. There are two possibilities:

n'	n''
1	0
0	2

---

*Reye, *Geometrie der Lage*, p. 18.

As an example of the first possibility, let the system be

$$\begin{cases} x+y=3, \\ x^2+y^2=5, \end{cases} \quad (x,\ y) \equiv (1,\ 2),\ (2,\ 1).$$

In order that the second possibility be satisfied, a_1^2 must equal $2a_2$. As an example of the second possibility, let the system be

$$\begin{cases} x+y=6, \\ x^2+y^2=18, \end{cases} \quad (x,\ y) \equiv (3,\ 3),\ (3,\ 3).$$

c. $k=1$, $l=3$, and, therefore, $kl=3$. Here, $2n'+n''=3$. There are two possibilities:

n'	n''
1	1
0	3

If we solve the system, $\begin{cases} x+y=a_1, \\ x^3+y^3=a_3, \end{cases}$ we find $(x,\ y) \equiv (\infty,\ \infty)$,

$$\left(\frac{a_1}{2} \pm \frac{1}{2}\sqrt{\frac{4a_3-a_1^3}{3a_1}},\ \frac{a_1}{2} \mp \frac{1}{2}\sqrt{\frac{4a_3-a_1^3}{3a_1}}\right).$$

(1) If $a_1=3$, $a_3=9$; $(x,\ y) \equiv (\infty,\ \infty)$, $(1, 2)$, $(2, 1)$, an example of the first possibility. The graph of this system is Fig. 4.

(2) In order that the second possibility be satisfied, a_1^3 must equal $4a_3$ or $a_1=0$.

Let $a_1=6$, and $a_3=54$. Then $(x,\ y) \equiv (\infty,\ \infty)$, $(3,\ 3)$, $(3, 3)$. The graph of the system is Fig. 5.

d. $k=1$, $l=4$, and, therefore, $kl=4$. Here, $2n'+n''=4$. There are three possibilities:

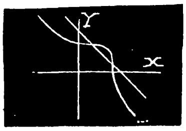

Fig. 4.

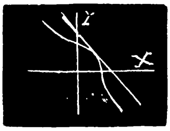

Fig. 5

n'	n''
2	0
1	2
0	4

Solving the system of equations $x+y=a_1$, $x^4+y^4=a_4$, we find (x, y)

$$\equiv \left(\frac{a_1 \pm \sqrt{[-3a_1^2 \mp 2\sqrt{(2a_1^4+2a_4)}]}}{2},\ \frac{a_1 \pm \sqrt{[-3a_1^2 \pm 2\sqrt{(2a_1^4+2a_4)}]}}{2}\right).$$

(1) If $a_1=3$, $a_4=17$; $(x, y)\equiv(1, 2)$, $(2, 1)$, $(3\pm\sqrt{-55}, 3\mp\sqrt{-55})$, an example of the first possibility. The graph of the system is the Fig. 6.

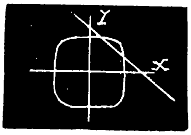

Fig. 6.

(2) For the second possibility, a_1^4 must equal $8a_4$. If $a_1=4$ and $a_4=32$, $(x, y)\equiv(2,2)$, $(2,2)$, $(2\pm2\sqrt{-6}, 2\mp2\sqrt{-6})$.

a_1 and a_4 must be so taken that $2a_1^4+2a_4$ is a perfect square. Thus, if $a_1=1$ and $a_4=8$, we have an example of the first possibility. If $a_1^4=-a_4$, we have an example also satisfying the first possibility, but since the graph of $x^4+y^4=-1$ is imaginary, all the roots are imaginary.

(3) The only example satisfying the third possibility is $x+y=0$, and $x^4+y^4=0$. Here, $(x, y)\equiv(0, 0)$, $(0, 0)$, $(0, 0)$, $(0, 0)$.

e. $k=1$, $l=5$, and, therefore, $kl=5$. Here, $2n'+n''=5$. There are three possibilities:

n'	n''
2	1
1	3
0	5

Solving the system of equations, $x+y=a_1$, $x^5+y^5=a_5$, we find $(x, y)\equiv(\infty, \infty)$,

$$\left[\tfrac{1}{2}\left(a_1 \pm\sqrt{-a_1^2\mp\tfrac{2}{5}\sqrt{[(5a_1^5+20a_5)/a_1)}}\right),\right.$$

$$\left.\tfrac{1}{2}\left((a_1\pm\sqrt{-5a_1^2\mp\tfrac{2}{5}\sqrt{[(5a_1^5+20a_5)/a_1]}}\right)\right].$$

(1) If $a=3$ and $a_5=33$, $(x, y)\equiv(1, 2)$, $(2, 1)$, $[\tfrac{1}{2}(3\pm\sqrt{-19}), \tfrac{1}{2}(3\mp\sqrt{-19})]$, (∞, ∞).

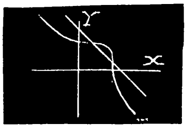

Fig. 7.

(2) For the second possibility, a_1^5 must equal $16a_5$, and $(5a_1^5+20a_5)/a_1$ must be a perfect square. If $a_1=4$ and $a_5=64$, $(x, y)\equiv(2, 2)$, $(2, 2)$, $(2\pm2\sqrt{-2}, 2\mp2\sqrt{-2})$, (∞, ∞).

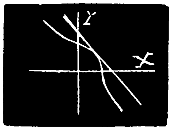

Fig. 8.

(3) The only example satisfying the third possibility is $x+y=0$, $x^5+x^5=0$. Here, $(x, y)\equiv(0, 0)$, $(0, 0)$, $(0, 0)$, $(0, 0)$, (∞, ∞).

f. $k=1$, $l=6$, and, therefore, $kl=6$. Here, $2n'+n''=6$. There are four possibilities:

n'	n''
3	0
2	2
1	4
0	6

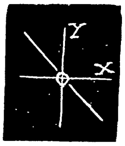

Fig. 9.

The solution of the system $x+y=a_1$, $x^6+y^6=a_6$ leads to a cubic for xy, As the values of x and y are quite complicated, we shall omit them in this case.

(1) If $a_1=3$ and $a_6=65$, $(x, y)\equiv(1, 2)$, $(2, 1)$, (α, β), (β, α), (γ, δ), (δ, γ), the last four sets being imaginary.

(2) If $a_1=2$ and $a_6=2$, $(x, y)\equiv(1, 1)$, $(1, 1)$, (α, β), (β, α), (γ, δ), (δ, γ).

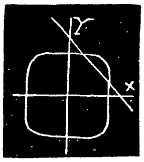

Fig. 10.

(3) The only equation satisfying the third possibility is $x+y=0$, $x^6+y^6=0$.

(4) The only equations satisfying the fourth possibility are $x+y=0$, $x^6+y^9=0$. There are six solutions each equal to $(x, y)\equiv(0, 0)$.

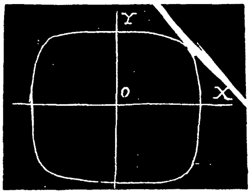

Fig. 11.

g. $k=2$, $l=3$, and, therefore, $kl=6$. Here, $2n'+n''=6$.

There are four possibilieies:

n'	n''
3	0
2	2
1	4
0	6

To solve the system $x^2+y^2=a_2$, $x^3+y^3=a_3$, requires the solution of a cubic in xy. If we let $y=rx$, we get the equation,

$$(a_2^3-a_3^2)(r^3+\frac{1}{r^3})-3a_3^2(r+\frac{1}{r})+2a_2^3=0, \text{ or } (a_2^3-a_3^2)u^3-3a_2^3u+2a_2^3=0,$$

where $u=(r+\dfrac{1}{r})$.

(1) If $a_2=13$ and $a_3=35$, $(x, y)\equiv(2, 3)$, $(3, 2)$, $[\frac{1}{2}(2\pm\sqrt{22}), \frac{1}{2}(2\mp\sqrt{22})]$, $[\frac{1}{2}(-7\pm i\sqrt{23}), \frac{1}{2}(-7\mp i\sqrt{23})]$.

(2) In order that the second possibility be satisfied, a_2^3 must equal $2a_3^2$.

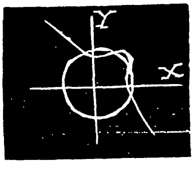

Fig. 12.

If $a_2=8$ and $a_3=16$, $(x, y)\equiv(2, 2)$, $(2, 2)$, $\{\pm[\sqrt{(4\pm2\sqrt{3})}+\sqrt{(\mp2\sqrt{3})}], \pm[\sqrt{(4\pm2\sqrt{3})}-\sqrt{(\mp2\sqrt{3})}]\}$.

(3, 4) In order that the third possibility be satisfied, a_2 and a_3 must both be equal to 0, and these are the only values of a_2 and a_3 which will also satisfy the fourth possibility.

Also solved by G. B. M. Zerr.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

120. Proposed by L. E. DICKSON, Ph. D., The University of Chicago.

Find the prime numbers p for which $x^2-pxz-px-z+p^2-3=0$ has more than two sets of positive integral solutions x, z, each $<p$.

Solution by H. S. VANDIVER, Bala, Pa.

About the time this problem was originally proposed, Professor Dickson wrote to me that he encountered it in determining those factors of $(p^2-1)^2$ which are of the form $1+px$, p prime. It will first be shown that the latter problem is equivalent to finding all the integers p, x, and z (p a prime) which satisfy

$$x^2-pxz-px-z+p^2-3=0 \ldots (1).$$

Let $(p^2-1)^2=(1+px)(1+py)$, where x and y are positive integers. Expanding and dividing by p,

$$p^3-2p=x+y+pxy.$$

Hence there is necessarily an integer $z>0$, such that $x+y=pz$. By substitution and division by p,

$$z(px+1)=x^2+p^2-2.$$

Setting $z+1$ for z we obtain (1), and the two problems are equivalent.

Solving (1) for p,

$$2p=x(z+1)\pm\sqrt{[x^2(z+3)(z-1)+4(z+3)]}.$$

For this to hold, there is necessarily a positive integer α such that

$$x^2(z+3)(z-1)+4(z+3)=\alpha^2,$$

subject to the condition that $x(z+1)\pm\alpha$ be double a prime.

Putting $z+3=v$, $\alpha^2-vx^2(v-4)=4v$.

Let $v=k^2u$, where u is an integer not divisible by a square other than unity, then

$$u\beta^2-x^2(k^2u-4)=4 \ldots (2),$$

where $\beta=\alpha/ku$, necessarily an integer.

The preceding analysis shows that (1) is soluble only for cases in which